# MATB44 Week 9 Notes

1. Non-Homogeneous Linear Systems:

- Has the form $\bar{x}' = A\bar{x} + \bar{g}$, where $\bar{g} \neq 0$.
- The general soln of a N-H linear system = The general soln of a H linear system + A particular soln of the N-H linear system.
- To find the particular soln, we will use variation of parameters.

E.g. 1 Solve $\bar{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

Soln:

Step 1: Solve the H linear system.

$$\bar{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \bar{x}$$

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0 \leftarrow \text{Characteristic Equation}$$

$(-2-r)^2 - 1 = 0$

$r^2 + 4r + 4 - 1 = 0$

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r_1 = -3, \ r_2 = -1$

$(A - rI)\bar{z} = 0 \leftarrow$ Eigenvalue Equation when $r=-3$

$$\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2(-3) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 + z_2 = 0$

$z_1 + z_2 = 0$ ← Redundant

$z_1 = -z_2$

Let $z_1 = 1$, $z_2 = -1$

$$\bar{z}^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

When $r=-1$

$$\begin{bmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-z_1 + z_2 = 0$

$z_1 - z_2 = 0$ ← Redundant

$z_1 = z_2$.

Let $z_1 = 1$, $z_2 = 1$.

$$\bar{z}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general soln to the H linear system is

$$\bar{x} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 2: Use variation of parameters to find a particular soln for the N-H linear system.

We have $u_1' e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + u_2' e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \bar{g}$
$= \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

and we want to solve for $u_1$ and $u_2$.

$$u_1' e^{-3t} + u_2' e^{-t} = 2e^{-t} \quad (1)$$
$$-u_1' e^{-3t} + u_2' e^{-t} = 3t \quad (2)$$

Do (1) + (2)

$$2u_2' e^{-t} = 2e^{-t} + 3t$$
$$u_2' = 1 + \frac{3te^{t}}{2}$$
$$u_2 = \int 1 + \frac{3te^{t}}{2}\, dt$$
$$= t + \frac{3}{2}(te^{t} - e^{t}) + c_2$$

$u_1' e^{-3t} + (1 + \frac{3te^{t}}{2})e^{-t} = 2e^{-t}$ ← Subbed $u_2'$ in (1).

$$u_1' e^{-3t} + e^{-t} + 3t/2 = 2e^{-t}$$
$$u_1' e^{-3t} = 2e^{-t} - e^{-t} - 3t/2$$
$$u_1' = (2e^{-t} - e^{-t} - 3t/2)e^{3t}$$
$$= 2e^{2t} - e^{2t} - \frac{3te^{3t}}{2}$$
$$u_1 = \int e^{2t} - \frac{3te^{3t}}{2}\, dt$$
$$= \frac{(3t-1)e^{3t}}{6} - \frac{e^{2t}}{2} + c_1$$

The particular soln is $U_1 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} + U_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

which equals to $\left(\frac{(3t-1)e^{3t}}{6} - \frac{e^{2t}}{2} + c_1\right)e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} +$

$\left(t + \frac{3}{2}(te^{t} - e^{t}) + c_2\right)e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$.

The general soln of the N-H linear system = The general soln of the H linear system + the particular soln of the N-H linear system.

E.g. 2 Solve $\bar{x}' = \begin{bmatrix}1 & -4\\4 & -7\end{bmatrix}\bar{x} + \begin{bmatrix}2e^{-3t}\\3te^{-3t}\end{bmatrix}$

Soln:

$\begin{vmatrix}1-r & -4\\4 & -7-r\end{vmatrix} = 0$

$(1-r)(-7-r)+16=0$

$-7-r+7r+r^2+16=0$

$r^2+6r+9=0$

$(r+3)^2=0$

$r=-3$

$(A-rI)\bar{z}=\bar{0}$

When $r=-3$

$\begin{bmatrix}1+3 & -4\\4 & -7+3\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$4z_1 - 4z_2 = 0$

$4z_1 - 4z_2 = 0$

$z_1 = z_2$

let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since we have repeated roots, we know that $\bar{x} = C_1 e^{rt} \overline{z^1} + C_2 (t e^{rt} \overline{z^1} + e^{rt} \bar{p})$, where $\bar{p}$ is the generalized eigenvector.

$$(A - rI)\bar{p} = \bar{z}$$

$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$4P_1 - 4P_2 = 1$

$P_1 - P_2 = 1/4$

let $P_1 = 0$. $P_2 = -1/4$

$$\bar{p} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$$

$$\bar{x} = C_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \left( t e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 0 \\ -1/4 \end{bmatrix} \right)$$

is the general soln of the H system.

To find the particular soln of the N-H system, we'll use variation of parameters.

$$U_1' e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2' \left( t e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 0 \\ -1/4 \end{bmatrix} \right) = \begin{bmatrix} 2e^{-3t} \\ 3te^{-3t} \end{bmatrix}$$

$$U_1' e^{-3t} + U_2' t e^{-3t} = 2e^{-3t}$$
$$U_1' e^{-3t} + U_2' t e^{-3t} - \frac{U_2' e^{-3t}}{4} = 3te^{-3t}$$

$$U_1' + U_2' t = 2 \qquad (1)$$
$$U_1' + U_2' t - \frac{U_2'}{4} = 3t \qquad (2)$$

Do (1) − (2)

$$\frac{U_2'}{4} = 2 - 3t$$
$$U_2' = 8 - 12t$$
$$U_2 = \int 8 - 12t \, dt$$
$$= 8t - 6t^2 + C_2$$

$$U_1' = 2 - U_2' t$$
$$= 2 - (8 - 12t)t$$
$$= 2 - 8t + 12t^2$$

$$U_1 = \int 2 - 8t + 12t^2 \, dt$$
$$= 2t - 4t^2 + 4t^3 + C_1$$

The particular soln of the N-H system is

$$(4t^3 - 4t^2 + 2t + C_1) e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} +$$
$$(-6t^2 + 8t + C_2)\left( te^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 0 \\ -1/4 \end{bmatrix} \right)$$

E.g. 3 Solve $\bar{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$

Soln:

$\begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0$

$(2-r)(-2-r)+3=0$

$-4-2r+2r+r^2+3=0$

$r^2-1=0$

$r^2=1$

$r=\pm 1$

$(A-rI)\bar{z}=\bar{0}$

when $r=1$

$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$z_1 - z_2 = 0$

$3z_1 - 3z_2 = 0$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$

$\overline{z'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $r=-1$

$$\begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$
$3z_1 - z_2 = 0$

$3z_1 = z_2$
Let $z_1 = 1$. $z_2 = 3$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The general soln of the H system is

$$\bar{x} = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Now, we will use variation of parameters to find the particular soln.

$$U_1' e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2' e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{t} \\ t \end{bmatrix}$$

$U_1' e^{t} + U_2' e^{-t} = e^{t}$ (1)
$U_1' e^{t} + 3U_2' e^{-t} = t$ (2)

Do (1) - (2)
$-2U_2' e^{-t} = e^{t} - t$
$U_2' = \frac{-1}{2}\left(e^{2t} - te^{t}\right)$

$$U_2 = \int -\frac{1}{2}\left(e^{2t} - te^{t}\right) dt$$

$$= \frac{-e^{t}\left(e^{t} - 2t + 2\right)}{4} + c_2$$

$$U_1' = 1 - U_2' e^{-2t}$$

$$= 1 - \left(\frac{-1}{2}\left(e^{2t} - te^{t}\right)\right) e^{-2t}$$

$$= 1 + \frac{1}{2}\left(1 - te^{-t}\right)$$

$$= \frac{3}{2} - \frac{1}{2} te^{-t}$$

$$U_1 = \int \frac{3}{2} - \frac{1}{2} te^{-t} \, dt$$

$$= \frac{3t}{2} + \frac{(t+1)e^{-t}}{2} + c_1$$

The particular soln is $U_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

E.g. 4 Solve $\bar{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$

Soln:

$$\begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix} = 0$$

$(2-r)(-2-r) + 5 = 0$

$-4 - 2r + 2r + r^2 + 5 = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$(A - rI)\bar{z} = \bar{0}$

wher $r = i$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2-i)z_1 - 5z_2 = 0$

$\frac{2-i}{5} z_1 = z_2$

Let $z_1 = 1$. $z_2 = \frac{2-i}{5}$

$$\bar{z'} = \begin{bmatrix} 1 \\ (2-i)/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2/5 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1/5 \end{bmatrix}$$

$\bar{x} = e^{rt}\bar{z}$

$e^{rt} = e^{it}$

$= \cos(t) + i\sin(t)$

$$(\cos(t) + i\sin(t)) \left( \begin{bmatrix} 1 \\ 2/5 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1/5 \end{bmatrix} \right)$$

$$= \cos(t) \begin{bmatrix} 1 \\ 2/5 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1/5 \end{bmatrix} +$$

$$i \left( \cos(t) \begin{bmatrix} 0 \\ -1/5 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 2/5 \end{bmatrix} \right)$$

The general soln of the H system is

$$c_1\left(\cos t\begin{bmatrix}1\\2/5\end{bmatrix} - \sin t\begin{bmatrix}0\\-1/5\end{bmatrix}\right) +$$

$$c_2\left(\cos t\begin{bmatrix}0\\-1/5\end{bmatrix} + \sin t\begin{bmatrix}1\\2/5\end{bmatrix}\right)$$

We can find the particular soln of the N-H system using variation of parameters.

$$u_1'\left(\cos t\begin{bmatrix}1\\2/5\end{bmatrix} - \sin t\begin{bmatrix}0\\-1/5\end{bmatrix}\right) +$$

$$u_2'\left(\cos t\begin{bmatrix}0\\-1/5\end{bmatrix} + \sin t\begin{bmatrix}1\\2/5\end{bmatrix}\right) = \begin{bmatrix}-\cos t\\\sin t\end{bmatrix}$$

$$u_1'\cos t + u_2'\sin t = -\cos t \quad (1)$$

$$u_1'\frac{2\cos t}{5} + u_1'\frac{\sin t}{5} - \frac{u_2'\cos t}{5} + \frac{u_2' 2\sin t}{5} = \sin t \quad (2)$$

From (1), $u_1' = -1 - \dfrac{u_2'\sin t}{\cos t}$

Plug into (2)

$$\left(-1 - \frac{u_2'\sin t}{\cos t}\right)(2\cos t) + \left(-1 - \frac{u_2'\sin t}{\cos t}\right)\sin t$$

$$- u_2'\cos t + 2u_2'\sin t = 5\sin t$$

$$-2\cos t - \cancel{2U_2'\sin t} - \sin t - U_2'\frac{\sin^2 t}{\cos t}$$

$$-U_2'\cos t + \cancel{2U_2'\sin t} = 5\sin t$$

$$-2\cos^2 t - \sin t\cos t - U_2'\sin^2 t - U_2'\cos^2 t = 5\sin t\cos t$$
$$-2\cos^2 t - U_2'(\sin^2 t + \cos^2 t) = 6\sin t\cos t$$
$$-2\cos^2 t - U_2' = 6\sin t\cos t$$
$$-U_2' = 6\sin t\cos t + 2\cos^2 t$$
$$U_2' = -6\sin t\cos t - 2\cos^2 t$$
$$U_2 = -\int 6\sin t\cos t + 2\cos^2 t\,dt$$
$$3\cos^2(t) + t + \frac{\sin(2t)}{2} + C_2$$

$$U_1' = -1 - \frac{U_2'\sin t}{\cos t}$$
$$= -1 + \frac{(6\sin t\cos t + 2\cos^2 t)\sin t}{\cos t}$$
$$= -1 + 6\sin^2 t + 2\sin t\cos t$$

$$U_1 = \int -1 + 6\sin^2 t + 2\sin t\cos t\,dt$$
$$= 3\left(t - \frac{\sin(2t)}{2}\right) - \cos^2(t) - t + C_1$$

The particular soln is $U_1\left(\cos t\begin{bmatrix}1\\2/5\end{bmatrix} - \sin t\begin{bmatrix}0\\-1/5\end{bmatrix}\right)$

$+ U_2\left(\cos t\begin{bmatrix}0\\-1/5\end{bmatrix} + \sin t\begin{bmatrix}1\\2/5\end{bmatrix}\right).$

**E.g. 5** Solve $\bar{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$

Soln:

$$\begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = 0$$

$(1-r)(-2-r) - 4 = 0$

$-2 - r + 2r + r^2 - 4 = 0$

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r_1 = -3, \ r_2 = 2$

$(A - rI)\bar{z} = \bar{0}$

When $r = -3$

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 + z_2 = 0$

$4z_1 = -z_2$

Let $z_1 = 1$. $z_2 = -4$.

$$\bar{z^1} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

When $r = 2$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-z_1 + z_2 = 0$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$.

$$\bar{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general soln of the H system is $c_1 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} +$

$c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We will use variation of parameters to find the particular soln.

$$u_1' e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + u_2' e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$$

$$u_1' e^{-3t} + u_2' e^{2t} = e^{-2t} \qquad (1)$$
$$-4u_1' e^{-3t} + u_2' e^{2t} = -2e^{t} \qquad (2)$$

Do (1) − (2)

$$5u_1' e^{-3t} = e^{-2t} + 2e^{t}$$
$$u_1' = \frac{e^{t} + 2e^{4t}}{5}$$
$$u_1 = \int \frac{e^{t} + 2e^{4t}}{5}\, dt$$
$$= \frac{e^{4t} + 2e^{t}}{10} + c_1$$

$$u_2' = e^{-4t} - u_1' e^{-5t}$$
$$= e^{-4t} - \left(\frac{e^{t} + 2e^{4t}}{5}\right) e^{-5t}$$
$$= e^{-4t} - \frac{e^{-4t}}{5} - \frac{2e^{-t}}{5}$$
$$= \frac{4e^{-4t}}{5} - \frac{2e^{-t}}{5}$$

$$u_2 = \int \frac{4e^{-4t}}{5} - \frac{2e^{-t}}{5}\, dt$$

$$= \frac{2e^{-t}}{5} - \frac{e^{-4t}}{5} + c_2$$

## 2. 3x3 Linear Systems:

- We will only be dealing with homogeneous systems if A is a 3 by 3 matrix.
- Superposition of Homogeneous systems:

If $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ are solns, then $c_1\bar{x}^1 \pm c_2\bar{x}^2 \pm \ldots \pm c_n\bar{x}^n$ is also a soln.

Since we are only dealing with 2x2 or 3x3 coefficient matrices, we can tailor the definition.

For 2x2: If $\overline{x^1}$ and $\overline{x^2}$ are solns, then $c_1\overline{x^1} \pm c_2\overline{x^2}$ is also a soln.

For 3x3: If $\overline{x^1}$ and $\overline{x^2}$ and $\overline{x^3}$ are solns, then $c_1\overline{x^1} \pm c_2\overline{x^2} \pm c_3\overline{x^3}$ is also a soln.

- Let $a = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$

$$|a| = A\begin{vmatrix} E & F \\ H & I \end{vmatrix} - B\begin{vmatrix} D & F \\ G & I \end{vmatrix} + C\begin{vmatrix} D & E \\ G & H \end{vmatrix}$$
$$= A(EI - HF) - B(DI - GF) + C(DH - GE)$$

E.g. 6 Let $A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$. Find $|A|$

Soln:

$$|A| = 6(-2\cdot 7 - 8\cdot 5) - 1(4\cdot 7 - 2\cdot 5) + 1(4\cdot 8 - (-2)\cdot 2)$$
$$= 6(-14-40) - (28-10) + (32+4)$$
$$= -324 - 18 + 36$$
$$= -306$$

- Let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $\bar{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$, where $\bar{x}$ $\bar{y}$ and $\bar{z}$ are functions of $t$.

If $c_1\bar{x} + c_2\bar{y} + c_3\bar{z} = 0$ for all $t$, then they are linearly dependent.

To know if $\bar{x}$, $\bar{y}$, and $\bar{z}$ are linearly dependent or not, we can use the Wronksian.

$$w = \begin{vmatrix} x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \\ x_3(t) & y_3(t) & z_3(t) \end{vmatrix}$$

$\bar{x}$, $\bar{y}$, and $\bar{z}$ are linearly dependent iff $w[x,y,z] = 0$.

- The general soln for homogeneous systems for a 3x3 coefficient matrix is $c_1\overline{x^1}(t) + c_2\overline{x^2}(t) + c_3\overline{x^3}(t)$, where $\overline{x^1}$, $\overline{x^2}$ and $\overline{x^3}$ are linearly independent solns.

- Since we are dealing with 3 x 3 coefficient matrices, we will be dealing with cubic functions. Hence, we need to know how to factor them. There are 2 techniques:

1. Factor By Grouping:

- Doesn't always work.
- If possible, factor out the greatest common factor out of each term. We get 2 cases:
    a) If you were able to factor out a variable, then you're left with a quadratic term.
    b) If you weren't able to factor out a variable, group the first 2 terms and last 2 terms together and then pull out common factors, if possible.
- If you can't factor out anything, group the first 2 terms and last 2 terms together and then pull out common factors, if possible.

E.g. 7 Factor $8x^3 + 4x^2 + 2x$.

Soln:
We see that $2x$ is a common factor for each term, so we pull it out first.

$2x\underbrace{(4x^2 + 2x + 1)}$

We know how to factor this (Quadratic Formula)

E.g. 8 Factor $x^3 + 7x^2 + 2x + 14$

Soln:
There's no greatest common factor here, so we just group the first 2 terms and the last 2 terms.

$(x^3 + 7x^2) + (2x + 14)$

$= x^2(x+7) + 2(x+7) = (x^2+2)(x+7)$

## 2. Factoring Using The Rational Root Thm:

- Works as long as the coefficients $a_0, a_1, a_2$ and $a_3$ are rational numbers.
- The possible roots are $\pm \frac{\text{factors of } a_0}{\text{factors of } a_3}$.

E.g. 9 Factor $x^3 + 5x^2 - 2x - 24$

Soln:

We can't factor by grouping.
The factors of $-24$ are $\pm 1, 2, 3, 4, 6, 8, 12, 24$.
The factors of 1 are $\pm 1$.
Therefore, the possible roots of this function are $\pm 1, 2, 3, 4, 6, 8, 12, 24$.

Try $x = 2$.
$8 + 20 - 4 - 24 = 0$.
This means that $x-2$ is a factor.
Now, we do polynomial division to find the other factors.

$$
\begin{array}{r|l}
 & x^2 + 7x + 12 \\
\hline
x-2 & x^3 + 5x^2 - 2x - 24 \\
 & -(x^3 - 2x^2) \\
\hline
 & \quad 7x^2 - 2x - 24 \\
 & \quad -(7x^2 - 14x) \\
\hline
 & \qquad 12x - 24 \\
 & \qquad -(12x - 24) \\
\hline
 & \qquad\quad 0
\end{array}
$$

Hence, $(x-2)(x^2 + 7x + 12) = x^3 + 5x^2 - 2x - 24$.

Now we'll solve H systems where the coefficient matrix is 3x3.

E.g. 10 Solve $\bar{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 0-r & 1 & 1 \\ 1 & 0-r & 1 \\ 1 & 1 & 0-r \end{vmatrix} = 0$$

$-r(r^2-1) - (-r-1) + (1+r) = 0$

$-r^3 + r + r + 1 + r + 1 = 0$

$-r^3 + 3r + 2 = 0$

$-r^3 + 2r + r + 2 = 0$

$(-r^3 + r) + (2r + 2) = 0$

$-r(r^2-1) + 2(r+1) = 0$

$-r(r-1)(r+1) + 2(r+1) = 0$

$(r+1)(-r(r-1) + 2) = 0$

$(r+1)(-r^2 + r + 2) = 0$

$(r+1)(r^2 - r - 2) = 0$

$(r+1)(r-2)(r+1) = 0$

$r_1 = -1, \ r_2 = 2, \ r_3 = -1$

$(A - rI)\bar{z} = \bar{0}$

When $r = -1$

$$\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} z_1 + z_2 + z_3 = 0 \\ z_1 + z_2 + z_3 = 0 \\ z_1 + z_2 + z_3 = 0 \end{matrix}\right\} \text{Redundant}$$

$z_3 = -z_1 - z_2$

Let $z_1 = 1$, $z_2 = 0$. Then, $z_3 = -1$

Let $z_1 = 0$, $z_2 = 1$. Then, $z_3 = -1$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \overline{z^2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Note: If we use this method, i.e. $z_1 = 0$ and $z_2 = 1$ and vice versa, then $\overline{z^1}$ and $\overline{z^2}$ will always be linearly independent.

When $r = 2$

$$\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2z_1 + z_2 + z_3 = 0$ (1)

$z_1 - 2z_2 + z_3 = 0$ (2)

$z_1 + z_2 - 2z_3 = 0$ (3)

If you do (2) − (3), eqn (2) becomes $-3z_2 + 3z_3 = 0$.

If you do $2 \cdot$ (3) + (1), eqn (3) becomes $3z_2 - 3z_3 = 0$.

I.e. Now we have

$$-2z_1 + z_2 + z_3 = 0$$
$$-3z_2 + 3z_3 = 0$$
$$3z_2 - 3z_3 = 0 \leftarrow \text{Redundant}$$

$$-3z_2 = -3z_3$$
$$z_2 = z_3$$

$$-2z_1 + 2z_2 = 0$$
$$-2z_1 = -2z_2$$
$$z_1 = z_2$$

$$z_1 = z_2 = z_3$$

let $z_1 = 1$. $z_2 = 1$, $z_3 = 1$

$$\overline{z^3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The general soln is $\overline{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$+ c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The Wronksion of $\overline{z^1}$, $\overline{z^2}$, and $\overline{z^3}$ is

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$$

$= 1(1+1) - 0 + 1(0-(-1))$
$= 2+1$
$= 3$
$\neq 0$

Hence, $\overline{z^1}$, $\overline{z^2}$, and $\overline{z^3}$ are linearly independent.
Furthermore, recall Abel's Formula.

When solving homogeneous linear systems with a 3x3 coefficient matrix A, Abel's Formula is:

$W = c' \cdot e^{-\int \text{Sum of main diagonal of A } dt}$

E.g. In this example, $W = c' \cdot e^{-\int 0+0+0 \, dt}$
$= c'$.

Since $W=3$, $c'=3$.

E.g. 11 Solve $\bar{x}' = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & -1 & -1 \\ 1 & 1-r & 0 \\ 3 & 0 & 1-r \end{vmatrix} = 0$$

$(1-r)(1-r)^2 - (-1)(1-r) - 1(0-3(1-r)) = 0$

$(1-r)^3 + 1-r + 3(1-r) = 0$

$(1-r)^3 + 4(1-r) = 0$

$(1-r)((1-r)^2+4) = 0$

$(1-r)(r^2-2r+5) = 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

$r_1 = 1, \; r_2 = 1+2i, \; r_3 = 1-2i$

$(A - rI)\bar{z} = \bar{0}$

when $r=1$

$$\begin{bmatrix} 1-1 & -1 & -1 \\ 1 & 1-1 & 0 \\ 3 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-z_2 - z_3 = 0$

$z_1 = 0$

$3z_1 = 0 \leftarrow$ Redundant

$$\left.\begin{array}{l} z_1 = 0 \\ z_2 = -z_3 \\ \text{let } z_2 = 1. \; z_3 = -1 \end{array}\right\} \overline{z^1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

When $r = 1+2i$

$$\begin{bmatrix} 1-(1+2i) & -1 & -1 \\ 1 & 1-(1+2i) & 0 \\ 3 & 0 & 1-(1+2i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -2iz_1 - z_2 - z_3 &= 0 \quad (1) \\ z_1 - 2iz_2 &= 0 \quad (2) \\ 3z_1 - 2iz_3 &= 0 \quad (3) \end{aligned}$$

If we add (2) and (3), we get $4z_1 - 2iz_2 - 2iz_3$ which equals to $2i \cdot (1)$, so it is redundant.

$z_1 = 2iz_2$

$-2i(2iz_2) - z_2 - z_3 = 0$

$3z_2 = z_3$

let $z_2 = 1$. $z_1 = 2i$, $z_3 = 3$

$$\overline{z^2} = \begin{bmatrix} 2i \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$e^{(1+2i)t}\left(\begin{bmatrix}0\\1\\3\end{bmatrix}+i\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$= e^{t}\cdot e^{i(2t)}\left(\begin{bmatrix}0\\1\\3\end{bmatrix}+i\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$= e^{t}\left(\cos(2t)+i\sin(2t)\right)\left(\begin{bmatrix}0\\1\\3\end{bmatrix}+i\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$= e^{t}\left(\cos(2t)\begin{bmatrix}0\\1\\3\end{bmatrix}-\sin(2t)\begin{bmatrix}2\\0\\0\end{bmatrix}\right)+$$

$$e^{t}\cdot i\left(\sin(2t)\begin{bmatrix}0\\1\\3\end{bmatrix}+\cos(2t)\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$\overline{x} = C_1 e^{t}\begin{bmatrix}0\\1\\-1\end{bmatrix}+e^{t}\cdot C_2\left(\cos(2t)\begin{bmatrix}0\\1\\3\end{bmatrix}-\sin(2t)\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$+C_3 e^{t}\left(\sin(2t)\begin{bmatrix}0\\1\\3\end{bmatrix}+\cos(2t)\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

E.g. 12 Solve $\bar{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{vmatrix} = 0$$

$(1-r)\left[(2-r)(1-r)-1\right] - (1-r-2) + 2(1-2(2-r)) = 0$

$(1-r)(2-2r-r+r^2-1) - (-r-1) + 2(1-4+2r) = 0$

$2-2r-2r+2r^2-r+r^2+r^2-r^3-1+r+r+1+4r-6=0$

$-r^3+4r^2+r-4=0$

$(-r^3+4r^2)+(r-4)=0$

$-r^2(r-4)+(r-4)=0$

$(r-4)(-r^2+1)=0$

$(r-4)(r^2-1)=0$

$(r-4)(r+1)(r-1)=0$

$r_1=4,\ r_2=-1,\ r_3=1$

$(A-rI)\bar{z}=\bar{0}$

When $r=4$

$$\begin{bmatrix} 1-4 & 1 & 2 \\ 1 & 2-4 & 1 \\ 2 & 1 & 1-4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3z_1 + z_2 + 2z_3 = 0$ (1)

$z_1 - 2z_2 + z_3 = 0$ (2)

$2z_1 + z_2 - 3z_3 = 0$ (3)

If you do (2) + (3), you get $3z_1 - z_2 - 2z_3 = 0$, which is $-1 \cdot (1)$. Hence, (3) is redundant.

$z_2 = 3z_1 - 2z_3$
$z_1 - 2(3z_1 - 2z_3) + z_3 = 0$
$z_1 - 6z_1 + 4z_3 + z_3 = 0$
$-5z_1 + 5z_3 = 0$
$-5z_1 = -5z_3$
$z_1 = z_3$
$z_2 = z_1$
Let $z_1 = 1$. $z_2 = 1$. $z_3 = 1$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When $r = -1$

$$\begin{bmatrix} 1-(-1) & 1 & 2 \\ 1 & 2-(-1) & 1 \\ 2 & 1 & 1-(-1) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2z_1 + z_2 + 2z_3 = 0$
$z_1 + 3z_2 + z_3 = 0$
$2z_1 + z_2 + 2z_3 = 0$ ← Redundant

$z_2 = -2z_1 - 2z_3$
$z_1 + 3(-2z_1 - 2z_3) + z_3 = 0$
$z_1 - 6z_1 - 6z_3 + z_3 = 0$
$-5z_1 - 5z_3 = 0$
$-5z_1 = 5z_3$
$-z_1 = z_3$

$$
\begin{aligned}
z_2 &= -2(z_1 + z_3) \\
&= -2(z_1 - z_1) \\
&= 0
\end{aligned}
$$

Let $z_1 = 1$. $z_3 = -1$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

When $r = 1$

$$\begin{bmatrix} 1-1 & 1 & 2 \\ 1 & 2-1 & 1 \\ 2 & 1 & 1-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
z_2 + 2z_3 &= 0 \\
z_1 + z_2 + z_3 &= 0 \\
2z_1 + z_2 \quad &= 0
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= -2z_3 \\
z_2 &= -2z_1 \\
z_1 &= z_3
\end{aligned}
$$

Let $z_1 = 1$. $z_2 = -2$, $z_3 = 1$

$$\overline{z^3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\bar{x} = C_1 e^{4t}\begin{bmatrix}1\\1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix} + C_3 e^{t}\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

**E.g. 13** Solve $\bar{x}' = \begin{bmatrix}3 & 2 & 4\\2 & 0 & 2\\4 & 2 & 3\end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix}3-r & 2 & 4\\2 & -r & 2\\4 & 2 & 3-r\end{vmatrix} = 0$$

$(3-r)((-r)(3-r)-4) - 2(2(3-r)-8) + 4(4+4r) = 0$
$(3-r)(-3r+r^2-4) - 2(6-2r-8) + 4(4+4r) = 0$
$-9r+3r^2+3r^2-r^3-12+4r-12+4r+16+16+16r=0$
$-r^3+6r^2+15r+8=0$

We need to use the Rational Root Theorem to find the factors.

Factors of 8: $\pm 1, 2, 4, 8$
Factors of -1: $\pm 1$

Try $r=-1$: $-(-1)^3+6(-1)^2+15(-1)+8$
$= 1+6-15+8$
$= 0$

$-1$ is a root. Hence, $r+1$ is a factor.

$$
\begin{array}{r}
-r^2 + 7r + 8 \\
r+1 \overline{\big) -r^3 + 6r^2 + 15r + 8} \\
-(-r^3 - r^2) \\
\hline
7r^2 + 15r + 8 \\
-(7r^2 + 7r) \\
\hline
8r + 8 \\
-(8r + 8) \\
\hline
0
\end{array}
$$

$$(r+1)(-r^2+7r+8) = -r^3 + 6r^2 + 15r + 8$$

$$\downarrow$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{49 + 32}}{-2}$$

$$= \frac{-7 \pm 9}{-2}$$

$$= -1 \text{ or } 8$$

Hence, the roots are:
$r = -1$, $r = 8$

$(A - rI)\bar{z} = \bar{0}$
when $r = -1$

$$\begin{bmatrix} 3+1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 3+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 4z_1 + 2z_2 + 4z_3 &= 0 \\ 2z_1 + z_2 + 2z_3 &= 0 \\ 4z_1 + 2z_2 + 4z_3 &= 0 \end{aligned}\right\} \text{Redundant}$$

$2z_1 + z_2 + 2z_3 = 0$

$z_2 = -2z_1 - 2z_3$

Let $z_1 = 0$ and $z_3 = 1$. $z_2 = -2$.

Lt $z_1 = 1$ and $z_3 = 0$. $z_2 = -2$.

$$\overline{z^1} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \quad \overline{z^2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

When $r = 8$

$$\begin{bmatrix} 3-8 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & 3-8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5z_1 + 2z_2 + 4z_3 = 0 \quad (1)$

$2z_1 - 8z_2 + 2z_3 = 0 \quad (2)$

$4z_1 + 2z_2 - 5z_3 = 0 \quad (3)$

If you do $-2 \cdot ((1) + (3))$, you get $2z_1 - 8z_2 + 2z_3 = 0$.
Hence, eqn 3 is redundant.

$-2z_2 = -5z_1 + 4z_3$

$2z_1 + 4(-5z_1 + 4z_3) + 2z_3 = 0$

$2z_1 - 20z_1 + 16z_3 + 2z_3 = 0$

$-18z_1 + 18z_3 = 0$

$-18z_1 = -18z_3$

$z_1 = z_3$

$2z_2 = z_1$

Let $z_1 = 1$. $z_2 = 2$, $z_3 = 1$.

$$\bar{z}^3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\bar{x} = C_1 e^{-t}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + C_3 e^{8t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

E.g. 14 Solve $\bar{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{vmatrix} = 0$$

$(1-r)\left[(1-r)^2 - 1\right] - 2(1-r) + (-2) = 0$

$(1-r)^3 - (1-r) - 2(1-r) - 2 = 0$

$(1-r)^3 - 3(1-r) - 2 = 0$

$-r^3 + 3r^2 - 3r + 1 - 3 + 3r - 2 = 0$

$-r^3 + 3r^2 - 4 = 0$

$-1$ is a root. $-(-1)^3 + 3(-1)^2 - 4$

$= 1 + 3 - 4$

$= 0$

Hence, $r+1$ is a factor.

$$
\begin{array}{r l}
 & -r^2 + 4r - 4 \\
r+1 & \overline{\big) -r^3 + 3r^2 + 0r - 4} \\
 & -(-r^3 - r^2) \\
 & \overline{\quad 4r^2 + 0r - 4} \\
 & -(4r^2 + 4r) \\
 & \overline{\quad -4r - 4} \\
 & -(-4r - 4) \\
 & \overline{\quad 0}
\end{array}
$$

$$(r+1)(-r^2+4r-4)=0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{16-16}}{-2}$$

$$= 2$$

$r_1 = -1$, $r_2 = 2$

$(A - rI)\bar{z} = \bar{0}$

When $r = -1$

$$\begin{bmatrix} 1+1 & 1 & 1 \\ 2 & 1+1 & -1 \\ 0 & -1 & 1+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

You can see that the $3^{rd}$ row of $A - rI$ equals to the first row - second row. Hence, it's redundant.

$$2z_1 + z_2 + z_3 = 0$$
$$2z_1 + 2z_2 - z_3 = 0$$

$$z_2 + z_3 = 2z_2 - z_3$$
$$2z_3 = z_2$$

$$2z_1 = -z_2 - z_3$$
$$= -3z_3$$
$$z_1 = \frac{-3z_3}{2}$$

let $z_2 = 1$. $z_3 = 2$, $z_1 = -3$

$$\overline{z^1} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

When $r = 2$

$$\begin{bmatrix} 1-2 & 1 & 1 \\ 2 & 1-2 & -1 \\ 0 & -1 & 1-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-z_1 + z_2 + z_3 = 0 \quad (1)$$
$$2z_1 - z_2 - z_3 = 0 \quad (2)$$
$$-z_2 - z_3 = 0 \quad (3)$$

If we do $-(2\cdot(1)+(2))$, we get (3), so (3) is redundant.

$-z_1+z_3=-2z_1+z_3$
$z_1=0$

$z_2=-z_3$
let $z_2=1$, $z_3=-1$

$$\overline{z^2}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

To find $\overline{z^3}$, we can use the method of generalized eigenvector.

$\bar{x}=c_1e^{r_1}\overline{z^1}+c_2e^{r_2}\overline{z^2}+c_3\left(te^{r_i}\overline{z^i}+\bar{p}e^{r_i}\right)$,
where $i=1$ or $2$ and $p$ is an unknown vector.

For this example, let $i=2$.
$(A-rI)\bar{p}=\bar{z}$

$$\begin{bmatrix}-1&1&1\\2&-1&-1\\0&-1&-1\end{bmatrix}\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$-p_1+p_2+p_3=0$
$2p_1-p_2-p_3=1$

$p_2=p_1-p_3$
$2p_1-p_1+p_3-p_3=1$
$p_1=1$

Let $p_2 = 0$. $p_3 = 1$.

$$\bar{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{x} = C_1 e^{-t} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} +$$

$$C_3 \left( t e^{2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$